CASA
Cyber Security in the Age of Large-Scale Adversaries

Quantum isomorphic strongly regular graphs from the E_8 root system

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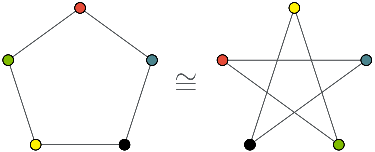
Graph isomorphism

Two graphs G and H are *isomorphic* if there exists a bijection σ such that

$$i \sim j \qquad \text{if and only if} \qquad \sigma(i) \sim \sigma(j)$$



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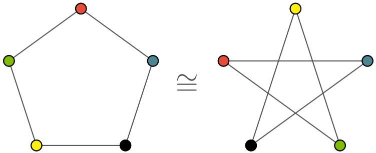

Theorem (Lovász)

It holds $G \cong H$ if and only if $|\mathrm{hom}(F, G)| = |\mathrm{hom}(F, H)|$ for all graphs F.

Quantum isomorphism

Two graphs G and H are called *quantum isomorphic* if

$$|\mathrm{hom}(F, G)| = |\mathrm{hom}(F, H)| \quad \text{for all } \textit{planar} \text{ graphs } F.$$

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Note

- G and H fractionally isomorphic $\iff$ $|\mathrm{hom}(T,G)| = |\mathrm{hom}(T,H)|$ for all trees T
- G and H cospectral $\iff$ $|\mathrm{hom}(C,G)| = |\mathrm{hom}(C,H)|$ for all cycles C

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Theorem (Atserias et al.)

There exist pairs of quantum isomorpic, but non-isomorphic graphs.

Quantum permutation matrices

A *quantum permutation matrix* $u = (u_{ij})_{i,j \in [m]}$ consists of $n \times n$ matrices u_{ij} fulfilling

$$u_{ij} = u_{ij}^* = u_{ij}^2 \quad \text{and} \quad \sum_k u_{ik} = \sum_k u_{ki} = \mathrm{Id}.$$

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$$m = 2 : \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} \qquad m = 4 : \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

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Theorem (Mančinska, Roberson)

Graphs G and H are quantum isomorphic if and only if there exists a quantum permutation matrix $u = (u_{ij})_{i\in V(G), j\in V(H)}$ such that $u_{ik}u_{jl} = 0$ if $i \sim j$ and $k \nsim l$ or vice versa.

Strongly regular graphs

A k-regular graph G is called *strongly regular* if there exists $\lambda, \mu \in \mathbb{N}_0$ such that

- adjacent vertices have λ common neighbors
- non-adjacent vertices have μ common neighbors

We then say that G is strongly regular with parameters (n, k, λ, μ).

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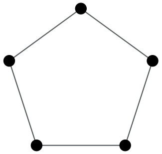

The 5-cycle is strongly regular with parameters $(5, 2, 0, 1)$

A strongly regular graph from the E_8 root system

The E_8 root system consists of the following 240 vectors in $\mathbb{R}^8$:

$$\pm e_i \pm e_j \text{ for } 1 \leq i < j \leq 8, \quad x = \frac{1}{2}(x_1, \ldots, x_8) \text{ for } x_i \in \{\pm 1\} \text{ and } \prod_{i=1}^{8} x_i = 1.$$

Let $G(E_8)$ be the orthogonality graph of the 120 lines spanned by the E_8 root system.

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Let $G(E_8)$ be the orthogonality graph of the 120 lines spanned by the E_8 root system.

- $G(E_8)$ is strongly regular with parameters $(120, 63, 30, 36)$
- It has independence number 8

A subgroup of the automorphism group of this graph

Define

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and } Y = XZ.$$

Lemma

For each $M := M_1 \otimes M_2 \otimes M_3$ with $M_i \in \{I, X, Y, Z\}$ the maps $\sigma_M : V(G(E_8)) \to V(G(E_8)),\ x \mapsto Mx$ are automorphisms of $G(E_8)$.

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We obtain a subgroup $L \cong \mathbb{Z}_2^6$ of $\mathrm{Aut}(G(E_8))$. The action of L on $V(G(E_8))$ has 15 orbits, partitioning the vertex set in 15 cliques of size 8.

$$\begin{aligned}
V_1 &: \quad e_1 \pm e_2, e_3 \pm e_4, e_5 \pm e_6, e_7 \pm e_8,\\
V_2 &: \quad e_1 \pm e_3, e_2 \pm e_4, e_5 \pm e_7, e_6 \pm e_8,\\
V_3 &: \quad e_1 \pm e_4, e_2 \pm e_3, e_5 \pm e_8, e_6 \pm e_7,\\
V_4 &: \quad e_1 \pm e_5, e_2 \pm e_6, e_3 \pm e_7, e_4 \pm e_8,\\
V_5 &: \quad e_1 \pm e_6, e_2 \pm e_5, e_3 \pm e_8, e_4 \pm e_7,\\
V_6 &: \quad e_1 \pm e_7, e_2 \pm e_8, e_3 \pm e_5, e_4 \pm e_6,\\
V_7 &: \quad e_1 \pm e_8, e_2 \pm e_7, e_3 \pm e_6, e_4 \pm e_5,
\end{aligned}$$

$$\begin{aligned}
V_8 &: \quad x_{\{1,2\}}, x_{\{3,4\}}, x_{\{5,6\}}, x_{\{7,8\}}, x_{\{1,4,6,8\}}, x_{\{2,3,6,8\}}, x_{\{2,4,5,8\}}, x_{\{2,4,6,7\}},\\
V_9 &: \quad x_{\{1,3\}}, x_{\{2,4\}}, x_{\{5,7\}}, x_{\{6,8\}}, x_{\{1,4,7,8\}}, x_{\{1,4,5,6\}}, x_{\{1,2,6,7\}}, x_{\{1,2,5,8\}},\\
V_{10} &: \quad x_{\{1,4\}}, x_{\{2,3\}}, x_{\{5,8\}}, x_{\{6,7\}}, x_{\{1,3,7,8\}}, x_{\{1,3,5,6\}}, x_{\{1,2,5,7\}}, x_{\{1,2,6,8\}},\\
V_{11} &: \quad x_{\{1,5\}}, x_{\{2,6\}}, x_{\{3,7\}}, x_{\{4,8\}}, x_{\{1,6,7,8\}}, x_{\{2,5,7,8\}}, x_{\{4,5,6,7\}}, x_{\{1,2,4,7\}},\\
V_{12} &: \quad x_{\{1,6\}}, x_{\{2,5\}}, x_{\{3,8\}}, x_{\{4,7\}}, x_{\{1,5,7,8\}}, x_{\{2,6,7,8\}}, x_{\{3,5,6,7\}}, x_{\{4,5,6,8\}},\\
V_{13} &: \quad x_{\{1,7\}}, x_{\{2,8\}}, x_{\{3,5\}}, x_{\{4,6\}}, x_{\{1,5,6,8\}}, x_{\{3,6,7,8\}}, x_{\{2,5,6,7\}}, x_{\{4,5,7,8\}},\\
V_{14} &: \quad x_{\{1,8\}}, x_{\{2,7\}}, x_{\{3,6\}}, x_{\{4,5\}}, x_{\{1,5,6,7\}}, x_{\{4,6,7,8\}}, x_{\{2,5,6,8\}}, x_{\{3,5,7,8\}},\\
V_{15} &: \quad x_{\varnothing}, x_{\{5,6,7,8\}}, x_{\{3,4,7,8\}}, x_{\{2,4,6,8\}}, x_{\{3,4,5,6\}}, x_{\{2,4,5,7\}}, x_{\{2,3,6,7\}}, x_{\{2,3,5,8\}}.
\end{aligned}$$

Construction of quantum permutation matrices

Denote by $P_x := \frac{1}{\|x\|^2} x x^*$ the rank-one projection associated to x.

Quantum permutation matrix

For every $i \in [15]$, choose $w_i \in V_i$. For $x, y \in V_i$, define

$$u^{(i)}_{x,y} = M_{xy} P_{w_i} M^*_{xy},$$

where $M_{xy} = M_1 \otimes M_2 \otimes M_3$ with $M_k \in \{I, X, Y, Z\}$ such that $M_{xy} x = y$.

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$$\begin{array}{c} \\ V_1 \\ V_2 \\ \vdots \\ \vdots \\ V_{15} \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & \cdots & \cdots & V_{15} \end{array} \\ \begin{pmatrix} u^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & u^{(2)} & 0 & \cdots & 0 \\ 0 & 0 & u^{(3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & u^{(15)} \end{pmatrix} \end{array}$$

The quantum isomorphism

Lemma

Let $k, s \in V_i$ and $l, t \in V_j$, $i \neq j$.

(i) For $k \sim l, s \sim t$, we have $u_{ks}^{(i)} u_{lt}^{(j)} = 0$ if and only if $\langle w_i, w_j \rangle = 0$.

(ii) For $k \sim l, s \nsim t$ or vice versa, we have $u_{ks}^{(i)} u_{lt}^{(j)} = 0$ if and only if $\langle w_i, w_j \rangle \neq 0$.

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Recall: $G \cong_q H \iff$ there exists a quantum permutation matrix $u = (u_{ij})_{i \in V(G), j \in V(H)}$ such that $u_{ik} u_{jl} = 0$ if $i \sim_G j$ and $k \nsim_H l$ or vice versa.

Quantum isomorphic graph

Let $\boldsymbol{w} = \{w_1, \ldots, w_{15}\}$ and define the graph $G^{\boldsymbol{w}}$ as follows. We let $V(G^{\boldsymbol{w}}) = V(G(E_8))$.

- If $s \in V_i$, $t \in V_j$, $\langle w_i, w_j \rangle \neq 0$, let $s \sim_{G^{\boldsymbol{w}}} t$ if and only if $s \sim_{G(E_8)} t$
- If $s \in V_i$, $t \in V_j$, $\langle w_i, w_j \rangle = 0$, let $s \sim_{G^{\boldsymbol{w}}} t$ if and only if $s \nsim_{G(E_8)} t$

Quantum isomorphic graphs

Theorem (S.)

The graphs $G(E_8)$ and $G^{\boldsymbol{w}}$ are quantum isomorphic, non-isomorphic strongly regular graphs.

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- It holds $G(E_8) \cong_q G^{\mathbf{w}}$, since the quantum permutation matrix we constructed fulfills $u_{ik}u_{jl} = 0$ if $i \sim_{G(E_8)} j$ and $k \nsim_{G^{\mathbf{w}}} l$ or vice versa.

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- It holds $G(E_8) \ncong G^{\mathbf{w}}$ since $\{w_1, \dots, w_{15}\}$ is an independent set of size 15 in $G^{\mathbf{w}}$. $G(E_8)$ has independence number 8.

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Lemma

The graphs $G^{\boldsymbol{w}_{(1)}}$ and $G^{\boldsymbol{w}_{(2)}}$ are isomorphic for any choice of $\boldsymbol{w}_{(1)}$, $\boldsymbol{w}_{(2)}$.

Another strongly regular graph with parameters $(120, 63, 30, 36)$

The graph $VO_6^+(2)$:

- Vertex set: $\mathbb{F}_2^6$
- Vertices x, y are adjacent if $Q(x+y) = 0$, where $Q(z) = z_1z_2 + z_3z_4 + z_5z_6$

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Another strongly regular graph

Define G_1 as follows.

- Vertex set: One orbit of the cliques of size 8 under $\mathbb{Z}_2^6 \times A_8$ in $VO_6^+(2)$
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Theorem (S.)

The complement of G_1 is isomorphic to $G^{\boldsymbol{w}}$.

Properties of the graphs

- $G(E_8)$ is distance-transitive, $\bar{G}_1$ is not
- We have $|\text{Aut}\, G(E_8)| = 348364800$, $\text{Aut}\, \bar{G}_1 = 1290240$
- $\bar{G}_1$ has independence number 15, the graph $G(E_8)$ has independence number 8

Godsil-McKay switching

Take a graph G, a partition $\pi = \{C_1, \ldots, C_k, D\}$ of $V(G)$. Suppose for $i, j \in [k]$, $v \in D$:

(i) any two vertices in C_i have the same number of neighbors in C_j,

(ii) the vertex v has either 0, $\frac{n_i}{2}$ or n_i neighbors in C_i, where $n_i := |C_i|$.

Define graph $G^{\pi,D}$: For each $v \in D$ and $i \in [k]$ such that v has $\frac{n_i}{2}$ neighbors in C_i, delete these $\frac{n_i}{2}$ edges and join v instead to the other $\frac{n_i}{2}$ vertices in C_i.

Theorem (Godsil, McKay)

The graphs G and $G^{\pi,D}$ are cospectral.

Godsil-McKay switching

Theorem (S.)

Assume $G_1 \cong_q G_2$ with quantum permutation matrix u of the form

$$\begin{array}{c} \\ V_1 \\ V_2 \\ \vdots \\ \vdots \\ V_m \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & \dots & \dots & V_m \end{array} \\ \begin{pmatrix} u^{(1)} & 0 & 0 & \dots & 0 \\ 0 & u^{(2)} & 0 & \dots & 0 \\ 0 & 0 & u^{(3)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & u^{(m)} \end{pmatrix} \end{array}$$

Let $\{S_1, \dots, S_{k+1}\}$ be a partition of $[m]$ and define a partition $\pi = \{C_1, \dots, C_k, D\}$ of the vertex set by $C_i := \cup_{s \in S_i} V_s$, $D := \cup_{s \in S_{k+1}} V_s$. If G_1 and G_2 fulfill the properties (i) and (ii) with respect to π, the graphs $G_1^{\pi,D}$ and $G_2^{\pi,D}$ are quantum isomorphic.

Summary

- We constructed two non-isomorphic strongly regular graphs whose homomorphism counts from all planar graphs coincide
- The quantum permutation matrix was constructed from a subgroup of $\mathrm{Aut}(G(E_8))$, related to Pauli matrices
- Using Godsil-McKay switching, one obtains more such pairs of quantum isomorphic strongly regular graphs

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Thank you for your attention!

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